Signed Spanning Pairs for Noncyclic Groups $\mathbb{Z}_2 \times \mathbb{Z}_{2k}$

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Abstract

Given a positive integer s, a noncyclic group of the form $G = \mathbb{Z}_2 \times \mathbb{Z}_{2k}$ for some positive integer k, and a subset $A \subset G$, let $[0, s]_{\pm} A$ denote the $[0, s]$-fold signed sumset of A. We are interested in the case where this signed sumset is the entire group G; in this case we say that A "spans" G. We investigate, for a given s, the maximum value of k such that a subset A with exactly two elements spans G.

This paper extends work done by Haesoo Park in 2020 on this same topic, providing a different proof for the case of odd values of s and a partial result for his conjecture for maximum values of k.

1 Introduction

We first define the main objects of our research, then introduce Park's previous results.

Definition 1. *Let s be a positive integer and let $A = \{a_1, a_2, \ldots, a_m\}$. The $[0, s]$-fold signed sumset of A is defined as*

$$[0, s]_{\pm} A = \{\lambda_1 a_1 + \lambda_2 a_2 + \cdots + \lambda_m a_m \mid |\lambda_1| + |\lambda_2| + \cdots + |\lambda_m| \in [0, s]\}.$$

Definition 2. *Let s be a positive integer, G be a group, and A a subset of G. Then A* spans *G if and only if $[0,s]_{\pm}A = G$. In this case we call A a* spanning set *of G, and denote by $\phi_{\pm}$ the size of the smallest spanning set of G for a given s:*

$$\phi_{\pm}(G,[0,s]) = \min\{|A| \mid [0,s]_{\pm}A = G\}.$$

Definition 3. *For a group of the form $G = \mathbb{Z}_2 \times \mathbb{Z}_{2k}$ for some k, we call the subset $\{0\} \times \mathbb{Z}_{2k}$ the* even *elements of G, and the subset $\{1\} \times \mathbb{Z}_{2k}$ the* odd *elements of G.*

We also include Park's results from [2]:

Theorem 4. *Given a positive integer s, let $k = \frac{s^2}{2}$ when s is even and $k = \frac{s^2-1}{2}$ when s is odd. Then $\phi_{\pm}(\mathbb{Z}_2 \times \mathbb{Z}_{2k},[0,s]) = 2$, where the spanning set of $\mathbb{Z}_2 \times \mathbb{Z}_{2k}$ is $\{(0,1),(1,s-1)\}$ when s is even and $\{(1,\frac{s-1}{2}),(1,\frac{s+1}{2})\}$ when s is odd.*

Conjecture 5. *The value of k found in the theorem above is the largest possible k for which $\phi_{\pm}(\mathbb{Z}_2 \times \mathbb{Z}_{2k},[0,s]) = 2$.*

Our work provides an alternative proof of Park's theorem with a different spanning set in the case where s is odd, and a result limiting potential counterexamples to Park's conjecture.

2 Main results

Theorem 6. *For any odd s, let $k = \frac{s^2-1}{2}$ and let $G = \mathbb{Z}_2 \times \mathbb{Z}_{2k} = \mathbb{Z}_2 \times \mathbb{Z}_{s^2-1}$. The pair $\{(0,x),\ (1,y)\}$ s-spans G, where x and y are defined by*

$$x = \begin{cases} \frac{s+1}{2}, & s \equiv 1 \bmod 4 \\ \frac{s-1}{2}, & s \equiv 3 \bmod 4 \end{cases} \qquad y = \begin{cases} \frac{s-1}{2}, & s \equiv 1 \bmod 4 \\ \frac{s+1}{2}, & s \equiv 3 \bmod 4. \end{cases}$$

Theorem 7. *If there is a counterexample to Park's conjecture, i.e. some $k > \frac{s^2}{2}$ such that*

$$\phi_{\pm}(\mathbb{Z}_2 \times \mathbb{Z}_{2k},[0,s]) = 2,$$

then the spanning pair must be of the form $\{(0,x),\ (1,y)\}$ for some $x,y \in \mathbb{Z}_{2k}$.

3 Methods

Theorem 6 [Proof]

We note some basic properties of x and y before continuing:

- x is always odd, and y is always even.
- $x + y = s$
- $4xy = 4 \cdot \frac{s^2-1}{4} = s^2 - 1$
- x and y are coprime because they differ by 1.

Given an arbitrary element $r \in G$, we show that there are coefficients λ_1, λ_2 that span r.

The span of $(0, x)$ will form a subgroup $H \leq G$ of order $\frac{s^2-1}{x} = 4y$, by our third identity above. This subgroup will have $\frac{|G|}{4y} = 2x$ corresponding cosets. The element r must lie in one of these cosets, so we will show that each of these cosets can be reached by some multiple of $(1, y)$.

We show that for each $\mu \in [0, 2x - 1]$, the product $\mu \cdot (1, y)$ reaches a different one of the $2x$ cosets of H. Because there are $2x$ different values μ, this implies that $\mu \cdot (1, y)$ reaches every coset of H.

To show that no two μ reach the same coset, we take two distinct $\mu_1, \mu_2 \in [0, 2x - 1]$, and assume without loss of generality that $\mu_1 > \mu_2$. Two elements are in the same coset of H if their difference is in H, so we prove our claim by showing that $(\mu_1 - \mu_2) \cdot (1, y) \notin \langle (0, x) \rangle$. Let $\mu' = \mu_1 - \mu_2 \in [1, 2x - 1]$. If $\mu' \cdot (1, y) \in \langle (0, x) \rangle$, then there is some c such that

$$\mu' \cdot (1, y) = c \cdot (0, x).$$

Because x and y are coprime, the only $\mu' \in [1, 2x - 1]$ to possibly satisfy the equation is x. However, because x is odd, the element $x \cdot (1, y)$ is also odd, and cannot be in the span of $(0, x)$.

The above implies that $(1, y)$ reaches all $2x$ cosets of $\langle (0, x) \rangle$. Therefore, for any coset of $\langle (0, x) \rangle$, there is some $\lambda_2 \in [0, 2x - 1]$ such that $\lambda_2 \cdot (1, y)$ is in the coset. Because each coset is of order $4y$, then if r is in the coset, there is some $\lambda_1 \in [-2y+1,\ 2y]$ such that $\lambda_1 \cdot (0,x) + \lambda_2 \cdot (1,y) = r$. We now turn our attention to the magnitude of the coefficients $\lambda_1,\ \lambda_2$.

To say that the pair spans r, we must have $\lambda_1,\ \lambda_2$ such that

$$\lambda_1 \cdot (0,x) + \lambda_2 \cdot (1,y) = r \qquad \text{and} \qquad |\lambda_1| + |\lambda_2| \in [0,s].$$

By their selection above, we know that $\lambda_1 \in [-2y+1,\ 2y]$ and $\lambda_2 \in [0, 2x-1]$, and therefore that

$$|\lambda_1| + |\lambda_2| \in [0,\ 2y+2x-1] = [0,\ 2s-1].$$

If $|\lambda_1|+|\lambda_2| \in [0,s]$, we are done. If, however, $|\lambda_1|+|\lambda_2| \in [s+1, 2s-1]$, we must find new $\lambda_1',\ \lambda_2'$ that yield the same element while remaining within the bounds. We address this case as the final component of the proof.

Choose $\lambda_1',\ \lambda_2'$ as follows:

$$\lambda_1' = \begin{cases} \lambda_1 - 2y, & \lambda_1 \geq 0 \\ \lambda_1 + 2y, & \lambda_1 < 0 \end{cases} \qquad \lambda_2' = \lambda_2 - 2x.$$

These definitions and our selection of $\lambda_1,\ \lambda_2$ imply that

$$|\lambda_1'| = 2y - |\lambda_1| \qquad \text{and} \qquad |\lambda_2'| = 2x - |\lambda_2|.$$

Consequently

$$\begin{aligned} |\lambda_1'| + |\lambda_2'| &= 2y - |\lambda_1| + 2x - |\lambda_2| \\ |\lambda_1'| + |\lambda_2'| &= 2(x+y) - (|\lambda_1| + |\lambda_2|) \\ |\lambda_1'| + |\lambda_2'| &= 2s - (|\lambda_1| + |\lambda_2|)\,. \end{aligned}$$

Recalling that $|\lambda_1| + |\lambda_2| \in [s+1,\ 2s-1]$, the above implies that $|\lambda_1'| + |\lambda_2'| \in [1, s-1]$, which is within the acceptable bounds for the coefficients.

Finally, we verify that $\lambda_1',\ \lambda_2'$ span the same element $r \in G$ as the original coefficients $\lambda_1,\ \lambda_2$.

If $\lambda_1 \geq 0$, and therefore $\lambda_1' = \lambda_1 - 2y$:

$$\begin{aligned}
\lambda_1' \cdot (0,x) + \lambda_2' \cdot (1,y) &= (\lambda_1 - 2y) \cdot (0,x) + (\lambda_2 - 2x) \cdot (1,y) \\
&= [\lambda_1 \cdot (0,x) + \lambda_2 \cdot (1,y)] - [2y \cdot (0,x) + 2x \cdot (1,y)] \\
&= r - (0, 4xy) \\
&= r - (0, s^2 - 1) \\
&= r - (0,0) \\
\lambda_1' \cdot (0,x) + \lambda_2' \cdot (1,y) &= r.
\end{aligned}$$

In the final case where $\lambda_1 < 0$ and $\lambda_1' = \lambda_1 + 2y$

$$\begin{aligned}
\lambda_1' \cdot (0,x) + \lambda_2' \cdot (1,y) &= (\lambda_1 + 2y) \cdot (0,x) + (\lambda_2 - 2x) \cdot (1,y) \\
&= [\lambda_1 \cdot (0,x) + \lambda_2 \cdot (1,y)] - 2y \cdot (0,x) + 2x \cdot (1,y) \\
&= r - (0, 2xy) + (0, 2xy) \\
\lambda_1' \cdot (0,x) + \lambda_2' \cdot (1,y) &= r.
\end{aligned}$$

Thus we have proven that λ_1', λ_2' are within the bounds for coefficients and span the arbitrary element $r \in G$. □

Theorem 7 [Proof]

Clearly no pair of the form $\{(0,x),\ (0,y)\}$ can span G. We now prove that no pair of the form $\{(1,x),(1,y)\}$ can span G for some $k > \frac{s^2}{2}$.

First, note that the parity of $|\lambda_1| + |\lambda_2|$ corresponds to the parity of the element spanned by the coefficients, $\lambda_1 \cdot (1,x) + \lambda_2 \cdot (1,y)$ — if one is even or odd, then the other must be even or odd, respectively. Due to this correspondence we call a coefficient pair (λ_1, λ_2) even or odd according to the parity of $|\lambda_1| + |\lambda_2|$.

We view these pairs of coefficients (λ_1, λ_2) as elements of the *two-dimensional integer lattice* $\mathbb{Z}^2([0,s])$, where $|\lambda_1| + |\lambda_2| \in [0,s]$. By the table found in [1, p. 28], we have

$$|\mathbb{Z}^2([0,s])| = 2s^2 + 2s + 1. \tag{1}$$

We can further divide this set of coefficient pairs into *layers* of the integer lattice: for some $h \in [0, s]$, its corresponding layer is defined as

$$\mathbb{Z}^2(h) = \{(\lambda_1, \lambda_2) \in \mathbb{Z}^2 \mid |\lambda_1| + |\lambda_2| = h\}.$$

We note another identity from [1, p. 28]:

$$|\mathbb{Z}^2(h)| = \begin{cases} 4h, & h \geq 1 \\ 1, & h = 0. \end{cases} \tag{2}$$

All coefficient pairs in $\mathbb{Z}^2(h)$ will be even if h is even and be odd if h is odd. By our observation above, their corresponding sum $\lambda_1 \cdot (1, x) + \lambda_2 \cdot (1, y)$ will then be even or odd, respectively. We will prove that, for $k > \frac{s^2}{2}$, there are either insufficient odd elements of $\mathbb{Z}^2([0, s])$ to span the odd elements of G, or insufficient even elements of $\mathbb{Z}^2([0, s])$ to span the even elements of G.

Let $E(s)$ denote the set of even coefficient pairs in $\mathbb{Z}^2([0, s])$ and let $O(s)$ denote the set of odd coefficient pairs in $\mathbb{Z}^2([0, s])$. Therefore by identity (1) we have

$$|E(s)| + |O(s)| = |\mathbb{Z}^2([0, s])| = 2s^2 + 2s + 1. \tag{3}$$

Now we calculate, using identity (2), the number of even coefficient pairs $|E(s)|$ for even s

$$\begin{aligned}
E(s) &= \mathbb{Z}^2(0) \cup \mathbb{Z}^2(2) \cup \cdots \cup \mathbb{Z}^2(s) \\
|E(s)| &= |\mathbb{Z}^2(0)| + |\mathbb{Z}^2(2)| + \cdots + |\mathbb{Z}^2(s)| \\
|E(s)| &= 1 + 4 \cdot 2 + \cdots + 4 \cdot s \\
|E(s)| &= 1 + 4 \cdot (2 + 4 + \cdots + s) \\
|E(s)| &= 1 + 8 \cdot (1 + 2 + \cdots + \frac{s}{2}) \\
|E(s)| &= 1 + 8 \cdot \frac{\frac{s}{2} \cdot (\frac{s}{2} + 1)}{2} \\
|E(s)| &= 1 + 8 \cdot \frac{s^2 + 2s}{8} \\
|E(s)| &= s^2 + 2s + 1.
\end{aligned}$$

By identity (3), this implies that $|O(s)| = s^2$ for even s. Now consider the quantity $|E(u)|$ for some odd u. Because u is odd, no elements of $\mathbb{Z}^2(u)$ are in $E(u)$, yielding

$$\begin{aligned} E(u) &= \mathbb{Z}^2(0) \cup \mathbb{Z}^2(2) \cup \cdots \cup \mathbb{Z}^2(u-1) \\ E(u) &= E(u-1) \\ |E(u)| &= (u-1)^2 + 2(u-1) + 1 \\ |E(u)| &= u^2. \end{aligned}$$

Because of the correspondence between even (odd) coefficient pairs and even (odd) spanned elements, the quantities $|E(s)|$ and $|O(s)|$ represent the maximum number of even and odd elements spanned by a pair of elements $\{(1, x),\ (1, y)\}$.

Consider the group $G = \mathbb{Z}_2 \times \mathbb{Z}_{2k}$ for any $k > \frac{s^2}{2}$. Because $2k > s^2$, there will be more than s^2 even and odd elements in G. If s is even, implying that $|O(s)| = s^2$, then the pair does not span enough odd elements to span all of G. If s is odd, and therefore $|E(s)| = s^2$, the pair does not span enough even elements to span all of G. Therefore no pair of the form $\{(1, x), (1, y)\}$ can span G for $k > \frac{s^2}{2}$.

No pair of even elements or pair of odd elements can span G for $k > \frac{s^2}{2}$, so any spanning pair for such a G must contain one even and one odd element: $\{(0, x), (1, y)\}$. □

4 Future work

A proof of **Conjecture 5** remains elusive, and the difference in proof of **Theorem 4** for even and odd values of s suggests that a proof of the conjecture may require two parts.

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References

[1] Béla Bajnok. *Additive Combinatorics: A Menu of Research Problems*. CRC Press, 2018.

[2] Haesoo Park. "Finding a Pair that Spans $\mathbb{Z}_2 \times \mathbb{Z}_{2k}$". In: *Research Papers in Mathematics* 23 (2020). Ed. by Béla Bajnok.